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RESEARCH MEMORANDUM

ON THE STRUCTURE OF CONSTRAINED EQUILIBRIA

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On the Structure of Constrained Equilibria * §

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Abstract

In this paper an exchange economy with price rigidities is considered. To obtain an equilibrium we allow for rationing on excess supply and on excess demand. Such an equilibrium is called a constrained equilibrium. It is shown that the rationing schemes allowed are very general. Several properties of the set of constrained equilibria are considered. More specifically, well-known properties like the existence of constrained equilibria with an unrationed numeraire commodity, supply constrained equilibria with at least one unrationed commodity, and demand constrained equilibria with at least one unrationed commodity follow as special cases from the theorems proved. Finally it is shown that the equilibrium correspondence, which assigns to each specification of initial endowments and set of admissible prices the set of all possible constrained equilibrium allocations, is upper semi-continuous. In order to prove these results a generalization of an existing continuity result of the budget correspondence is given.

1 Introduction

To prove the existence of a Walrasian equilibrium, prices have to be assumed to be completely flexible. If the constraints imposed by price rigidities are included in the set of admissible prices a Walrasian equilibrium price does not necessarily exist. To reach an equilibrium of an economy with price rigidities, quantity adjustments by means of rationing are made instead of price adjustments. If price rigidities and quantity rationing are included into a general equilibrium model, an equilibrium concept has to be defined which differs from the one used by Debreu (1959). The corresponding equilibria are called constrained equilibria. The constraints on the set of admissible prices are given by lower and upper bounds on the prices on each market and are based on Drèze (1975). Models with linked prices, like for example Kurz (1982), van der Laan (1984), and Dehez and Drèze (1984), are not considered in this paper. However, making some substitutions, the set of admissible prices considered in these papers is similar to the case where the set of admissible prices is obtained by specifying upper and lower bounds.

In Drèze's model the consumers express their rationed demand. This means that the excess demand of a consumer must satisfy the rationing constraints. Others, like Benassy (1975) and Malinvaud (1977) consider the effective demand of a consumer. In his effective demand of some commodity, the consumer does take into account the rationing constraints on all markets, except the market of that commodity. This kind of analysis will not be performed in this paper.

In Drèze (1975) the existence of a constrained equilibrium without rationing on the market of an a priori chosen commodity is shown. The condition that there is no rationing on the market of at least one commodity guarantees that the constrained equilibrium obtained is non-trivial. Moreover the condition guarantees that no problems arise with respect to the continuity of the budget correspondence of the consumers. In van der Laan (1980) and Kurz (1982) it has been remarked that supply rationing is observed more frequently in real world economies than demand rationing. In van der Laan (1982) the existence of a constrained equilibrium with supply rationing only and with at least one market without rationing has been proved, using the technique of simplicial approximation of equilibria. In van der Laan and Talman (1990) this result is proved without using the simplicial approximation technique.

In this paper the above results will be extended in two ways. First the rationing schemes allowed will be more general. Secondly it will be shown that a continuum of constrained equilibria exists, which is in accordance with the examples given in Böhm and Müller (1977) and with the results obtained in van der Laan and Talman (1990). A continuum of correspondences will be given such that each correspondence has a fixed point and each fixed point corresponds with a constrained equilibrium. It will be shown that without loss of generality the situation on a market, which is determined by the rationing scheme and the price, can be described by one parameter, which is called the pseudo-price. A pseudo-price determines either the level of demand rationing, or the price, or the level of supply rationing. It is shown in Section 3 that for every given value of a pseudo-price of an a priori chosen commodity, a constrained equilibrium exists. In order to give a proof of this result an extension of the proof of the continuity of the budget correspondence of Drèze (1975) is needed. This proof is given in Section 2.

In Section 4 it is shown that there exists a constrained equilibrium with every pseudo-price less than or equal to a value specified for each market, and with at least one pseudo-price equal to this value. Thereby the result that a constrained equilibrium with only supply rationing and at least one market without rationing exists, is generalized. Although van der Laan (1980) and Kurz (1982) argued that in real world economies supply rationing seems to occur more frequently than demand rationing, recent experiences in Eastern Europe show that also constrained equilibria with demand rationing are interesting. Extensions of the results in this way are given in Section 5. For an explicit modelling of demand rationing in a socialist economy, see Boycko (1992). In Section 5 it is shown that there exists a constrained equilibrium with every pseudo-price greater than or equal to a value specified for each market, and with at least one pseudo-price equal to this value. A special case is the existence of a constrained equilibrium with only demand rationing and at least one market without rationing. These results depend again on the proof of the continuity of the budget correspondence in Section 2. It is shown that using the correspondences of the Sections 3, 4, and 5, all kind of constrained equilibria can be obtained. Hence it is possible to compute an approximation of a constrained equilibrium satisfying any of the above mentioned properties.

Given the consumption sets and the preferences of the consumers, the equilibrium correspondence assigns to each specification of initial endowments and set of admissible prices the set of all possible constrained equilibrium allocations. In Section 6 it is shown that the equilibrium correspondence is upper semi-continuous. This is an extension of Theorem 1.4 of Laroque and Polemarchakis (1978). There it is shown that, in case the lower bound on the set of admissible prices is equal to the upper bound on the set of admissible prices, the correspondence which assigns to each specification of initial endowments and fixed price the set of constrained equilibria without rationing on an a priori chosen numeraire commodity, has a closed graph.

2 The Continuity of the Budget Correspondence in a Model with Price Rigidities

In this section a model is developed to deal with upper and lower bounds on the prices in an exchange economy defined by $E = (\{X^i, \succeq^i, w^i\}_{i=1}^m, P_{(\underline{p},\bar{p})})$. There are m consumers indexed $i = 1, \ldots, m$ and n commodities indexed $j = 1, \ldots, n$. Each consumer is defined by a consumption set X^i, a preference ordering $\succeq^i$ on X^i, and a vector of initial endowments w^i. The allocation of initial endowments is denoted by w, so $w = (w^1, \ldots, w^m)$. The set of admissible prices is denoted by $P_{(\underline{p},\bar{p})}$. In the following for $k \in \mathbb{N}$, I_k denotes the set of integers $\{1, \ldots, k\}$ and $\mathbf{R}^k_+$ denotes the non-negative orthant of the k-dimensional Euclidean space $\mathbf{R}^k$. If $x, y \in \mathbf{R}^k$ then $x \geq y$ means $x_j \geq y_j$, $\forall j \in I_k$, $x > y$ means $x \geq y$ and $\exists j \in I_k$ such that $x_j > y_j$, and $x \gg y$ means $x_j > y_j$, $\forall j \in I_k$. $\mathbf{R}^k_{++}$ denotes the set $\{x \in \mathbf{R}^k \mid x \gg 0\}$. If A is a subset of $\mathbf{R}^k$ then $\text{Int}(A)$ denotes the interior of A in $\mathbf{R}^k$. With respect to the economy E the following assumptions are made:

A1. X^i is a convex, closed, non-empty subset of $\mathbf{R}^n$, $X^i \subset \mathbf{R}^n_+$, and $X^i + \mathbf{R}^n_+ \subset X^i$.

A2. The preference ordering $\succeq^i$ on X^i is transitive, complete, continuous, strongly monotonic, and convex.

A3. The initial resources w^i are an element of $\text{Int}(X^i)$.

A4. The set of admissible prices is equal to

$$P_{(\underline{p},\bar{p})} = \left\{ p \in \mathbf{R}^n_+ \,\middle|\, \underline{p}_j \leq p_j \leq \bar{p}_j,\ \forall j \in I_n \right\},$$

for $\underline{p}, \bar{p} \in \mathbf{R}^n_{++}$ such that $\underline{p}_j \leq \bar{p}_j$, for all $j \in I_n$.

As has been shown in Debreu (1959) these assumptions imply that it is possible to represent the preferences of consumer $i \in I_m$ by a continuous quasi-concave utility function u^i from X^i into $\mathbf{R}$.

A vector l^i such that $l^i \in -\mathbf{R}^n_+$ denotes a constraint on the excess supply of consumer $i \in I_m$ and a vector L^i such that $L^i \in \mathbf{R}^n_+$ denotes a constraint on the excess demand of consumer $i \in I_m$. The constrained budget set of consumer $i \in I_m$ at price $p \in \mathbf{R}^n_{++}$, rationing scheme $(l^i, L^i) \in -\mathbf{R}^n_+ \times \mathbf{R}^n_+$, and initial endowments $w^i \in \text{Int}\,(X^i)$, is equal to

$$B^i\left(p, l^i, L^i, w^i\right) = \left\{ x^i \in X^i \,\middle|\, p \cdot x^i \leq p \cdot w^i,\ l^i \leq x^i - w^i \leq L^i \right\}.$$

The constrained budget set of consumer $i \in I_m$ is non-empty for all $p \in \mathbf{R}^n_{++}$, $l^i \in -\mathbf{R}^n_+$, $L^i \in \mathbf{R}^n_+$, and $w^i \in \text{Int}\,(X^i)$, because $w^i \in B^i\,(p, l^i, L^i, w^i)$.

The demand of consumer $i \in I_m$ at price $p \in \mathbf{R}^n_{++}$, rationing scheme $(l^i, L^i) \in -\mathbf{R}^n_+ \times \mathbf{R}^n_+$, and initial endowments $w^i \in \text{Int}\,(X^i)$, is denoted by $\delta^i\,(p, l^i, L^i, w^i)$ and is given by

$$\delta^i\left(p, l^i, L^i, w^i\right) = \left\{ x^i \in B^i\left(p, l^i, L^i, w^i\right) \,\middle|\, u^i(x^i) = \max_{y^i \in B^i(p, l^i, L^i, w^i)} u^i(y^i) \right\}.$$

The following definition of a constrained equilibrium is equal to the one given in Drèze (1975).

Definition 2.1 (Constrained Equilibrium)
A constrained equilibrium of the economy $E = (\{X^i, \succeq^i, w^i\}_{i=1}^m, P_{(\underline{p},\bar{p})})$ *is an element*

$$\left(x^{*1}, \ldots, x^{*m}, l^{*1}, \ldots, l^{*m}, L^{*1}, \ldots, L^{*m}, p^*\right)$$

of the set

$$\prod_{i=1}^{m} X^i \times \prod_{i=1}^{m} -\mathbb{R}_+^n \times \prod_{i=1}^{m} \mathbb{R}_+^n \times P_{(\underline{p},\bar{p})}$$

such that

1. $\forall i \in I_m : x^{*i} \in \delta^i \left(p^*, l^{*i}, L^{*i}, w^i\right);$
2. $\sum_{i=1}^m x^{*i} - \sum_{i=1}^m w^i = 0;$
3. $\forall j \in I_n : x_j^{*h} - w_j^h = L_j^{*h}$ *for some* $h \in I_m$ *implies* $x_j^{*i} - w_j^i > l_j^{*i}\ \forall i \in I_m$, *and* $x_j^{*h} - w_j^h = l_j^{*h}$ *for some* $h \in I_m$ *implies* $x_j^{*i} - w_j^i < L_j^{*i}\ \forall i \in I_m;$
4. $\forall j \in I_n : p_j^* < \bar{p}_j$ *implies* $L_j^{*i} > x_j^{*i} - w_j^i\ \forall i \in I_m$, *and* $p_j^* > \underline{p}_j$ *implies* $l_j^{*i} < x_j^{*i} - w_j^i\ \forall i \in I_m$.

In order to show the upper semi-continuity of the demand correspondence it is necessary to show that the constrained budget correspondence B^i from $\mathbb{R}_{++}^n \times -\mathbb{R}_+^n \times \mathbb{R}_+^n \times \text{Int}\left(X^i\right)$ into X^i is continuous for all $i \in I_m$. In Drèze (1975) $w^i \in \text{Int}\left(X^i\right)$ and $P_{(\underline{p},\bar{p})}$ are considered as given and it is shown that the correspondence $B^i\left(\cdot,\cdot,\cdot,w^i\right)$ is continuous at $\left(p, l^i, L^i\right) \in P_{(\underline{p},\bar{p})} \times -\mathbb{R}_+^n \times \mathbb{R}_+^n$ when for some $j \in I_n$, $p_j > 0$ and $l_j^i < 0$. This last condition is restrictive if one wants to show the existence of a continuum of constrained equilibria, especially if the rationing schemes are not uniform, as will be the case in the subsequent sections. In Drèze (1975) some prices are allowed to be zero. However, it turns out that if prices are all positive the budget correspondence is continuous, even if $l_j^i = 0$ for all $j \in I_n$. In order to show the upper semi-continuity of the equilibrium correspondence at initial endowments satisfying Assumption A3 and set of admissible prices satisfying Assumption A4, it is not enough to examine the continuity of the budget correspondence given $w^i \in \text{Int}\left(X^i\right)$ and $P_{(\underline{p},\bar{p})}$, but instead the more general case proved in Theorem 2.2 has to be considered.

Theorem 2.2
For $i \in I_m$, *if* X^i *is closed, convex,* $X^i \subset \mathbb{R}_+^n$, *and* $Int\left(X^i\right) \neq \emptyset$, *then* B^i *is a continuous correspondence from* $\mathbb{R}_{++}^n \times -\mathbb{R}_+^n \times \mathbb{R}_+^n \times Int\left(X^i\right)$ *into* X^i.
Proof
Let $\left(p, l^i, L^i, w^i\right) \in \mathbb{R}_{++}^n \times -\mathbb{R}_+^n \times \mathbb{R}_+^n \times \text{Int}\left(X^i\right)$.
The proof consists of two parts.

1. B^i is upper semi-continuous at $\left(p, l^i, L^i, w^i\right)$.
2. B^i is lower semi-continuous at $\left(p, l^i, L^i, w^i\right)$.

1. B^i *is upper semi-continuous at* $\left(p, l^i, L^i, w^i\right)$.
Clearly, $B^i\left(p, l^i, L^i, w^i\right)$ is non-empty, since $w^i \in B^i\left(p, l^i, L^i, w^i\right)$, and B^i is compact valued,

since

$$B^i\left(p,l^i,L^i,w^i\right)=\left\{x^i\in X^i\,\middle|\,p\cdot x^i\le p\cdot w^i,\ l^i\le x^i-w^i\le L^i\right\}$$

is closed and $x^i\in B^i(p,l^i,L^i,w^i)$ implies

$$\left\|x^i\right\|_\infty\le\max\left\{\frac{p\cdot w^i}{p_1},\ldots,\frac{p\cdot w^i}{p_n}\right\},$$

and therefore $B^i(p,l^i,L^i,w^i)$ is bounded. Let $((p^s,l^{i^s},L^{i^s},w^{i^s}))_{s\in\mathbf{N}}$ be a sequence of points of $\mathbf{R}^n_{++}\times-\mathbf{R}^n_+\times\mathbf{R}^n_+\times\mathrm{Int}(X^i)$ converging to (p,l^i,L^i,w^i) and let $(x^{i^s})_{s\in\mathbf{N}}$ be a sequence in X^i with $x^{i^s}\in B^i(p^s,l^{i^s},L^{i^s},w^{i^s})$. The sequence $(x^{i^s})_{s\in\mathbf{N}}$ is bounded since

$$0\le x^{i^s}\le\max\left\{\frac{p^s\cdot w^{i^s}}{p_1^s},\ldots,\frac{p^s\cdot w^{i^s}}{p_n^s}\right\}\to\max\left\{\frac{p\cdot w^i}{p_1},\ldots,\frac{p\cdot w^i}{p_n}\right\}.$$

Let $(x^{i^{s^t}})_{t\in\mathbf{N}}$ be a convergent subsequence of the sequence $(x^{i^s})_{s\in\mathbf{N}}$ with limit, say $\tilde{x}^i$. It will be shown that $\tilde{x}^i\in B^i(p,l^i,L^i,w^i)$ thereby showing the upper semi-continuity of B^i at (p,l^i,L^i,w^i). Clearly $\tilde{x}^i\in X^i$ since $x^{i^{s^t}}\in X^i$ and X^i is closed. Moreover

$$\begin{aligned}
p\cdot\tilde{x}^i &= \lim_{t\to\infty}p^{s^t}\cdot x^{i^{s^t}}\le\lim_{t\to\infty}p^{s^t}\cdot w^{i^{s^t}}=p\cdot w^i,\\
\tilde{x}^i-w^i &= \lim_{t\to\infty}(x^{i^{s^t}}-w^{i^{s^t}})\ge\lim_{t\to\infty}l^{i^{s^t}}=l^i,\\
\tilde{x}^i-w^i &= \lim_{t\to\infty}(x^{i^{s^t}}-w^{i^{s^t}})\le\lim_{t\to\infty}L^{i^{s^t}}=L^i.
\end{aligned}$$

2. B^i is lower semi-continuous at (p,l^i,L^i,w^i).
Let $((p^s,l^{i^s},L^{i^s},w^{i^s}))_{s\in\mathbf{N}}$ be a sequence of points of $\mathbf{R}^n_{++}\times-\mathbf{R}^n_+\times\mathbf{R}^n_+\times\mathrm{Int}(X^i)$. The correspondence B^i is lower semi-continuous at the point $(p,l^i,L^i,w^i)\in\mathbf{R}^n_{++}\times-\mathbf{R}^n_+\times\mathbf{R}^n_+\times\mathrm{Int}(X^i)$ if

$$\begin{aligned}
&\left(p^s,l^{i^s},L^{i^s},w^{i^s}\right)\to\left(p,l^i,L^i,w^i\right)\text{ and }x^i\in B^i\left(p,l^i,L^i,w^i\right)\\
&\Rightarrow\exists x^{i^s}\text{ such that }x^{i^s}\in B^i\left(p^s,l^{i^s},L^{i^s},w^{i^s}\right)\text{ and }x^{i^s}\to x^i.
\end{aligned}$$

Two subcases have to be considered.

2.1. $p\cdot(x^i-w^i)<0$.
2.2. $p\cdot(x^i-w^i)=0$.

2.1. $p\cdot(x^i-w^i)<0$.
Define the following sets

$$\begin{aligned}
J &= \left\{j\,\middle|\,x_j^i>w_j^i\right\},\\
K &= \left\{j\,\middle|\,x_j^i=w_j^i\right\},\\
L &= \left\{j\,\middle|\,x_j^i<w_j^i\right\}.
\end{aligned}$$

Since $w^i \in \mathrm{Int}\,(X^i)$ there exists an $\varepsilon > 0$ such that $\|y^i - w^i\|_\infty \leq \varepsilon$ implies $y^i \in X^i$. There exists an $s^1 \in \mathbb{N}$ such that for all $s > s^1$, $\|w^{i^s} - w^i\|_\infty \leq \varepsilon$. For $s > s^1$ define

$$\begin{array}{rcll} a_j^{i^s} & = & w_j^{i^s}, & j \in K, \\ a_j^{i^s} & = & \lambda^{i^s} x_j^i + (1 - \lambda^{i^s})\, w_j^{i^s}, & j \in J \cup L, \end{array}$$

where if $K = \emptyset$

$$\lambda^{i^s} = 1 \tag{1}$$

and if $K \neq \emptyset$

$$\lambda^{i^s} = \min\left\{ \frac{\varepsilon - |w_j^{i^s} - w_j^i|}{\varepsilon} \middle| j \in K \right\}. \tag{2}$$

It will be shown that $a^{i^s} \in X^i$ for all $s > s^1$ and that $a^{i^s} \to x^i$. Clearly, for all $s > s^1$ it holds that $0 \leq \lambda^{i^s} \leq 1$. If $\lambda^{i^s} = 1$ then by (1) and (2), $w_j^{i^s} = w_j^i$, $\forall j \in K$, and

$$\begin{array}{rcccccl} a_j^{i^s} & = & w_j^{i^s} & = & w_j^i & = & x_j^i, \quad \forall j \in K, \\ a_j^{i^s} & = & x_j^i, & & & & \quad \forall j \in J \cup L, \end{array}$$

and so $a^{i^s} \in X^i$.
If $\lambda^{i^s} < 1$ then define

$$b^{i^s} = \frac{a^{i^s} - \lambda^{i^s} x^i}{1 - \lambda^{i^s}}.$$

It will first be shown that $b^{i^s} \in X^i$.
Suppose $j \in K$, then

$$\left| b_j^{i^s} - w_j^i \right| = \left| \frac{w_j^{i^s} - \lambda^{i^s} x_j^i}{1 - \lambda^{i^s}} - w_j^i \right| = \frac{\left| w_j^{i^s} - w_j^i \right|}{1 - \lambda^{i^s}}.$$

If $w_j^{i^s} = w_j^i$ then

$$\left| b_j^{i^s} - w_j^i \right| = 0 < \varepsilon,$$

and if $w_j^{i^s} \neq w_j^i$ then

$$\left| b_j^{i^s} - w_j^i \right| = \frac{\left| w_j^{i^s} - w_j^i \right|}{1 - \min\left\{ \frac{\varepsilon - \left| w_k^{i^s} - w_k^i \right|}{\varepsilon} \middle| k \in K \right\}} \leq \frac{\left| w_j^{i^s} - w_j^i \right|}{1 - \frac{\varepsilon - \left| w_j^{i^s} - w_j^i \right|}{\varepsilon}} = \varepsilon.$$

Suppose $j \in J \cup L$, then

$$\left| b_j^{i^s} - w_j^i \right| = \left| \frac{\lambda^{i^s} x_j^i + (1 - \lambda^{i^s})\, w_j^{i^s} - \lambda^{i^s} x_j^i}{1 - \lambda^{i^s}} - w_j^i \right| = \left| w_j^{i^s} - w_j^i \right| \leq \varepsilon.$$

Hence, since $\|b^{i^s} - w^i\|_\infty \leq \varepsilon$, $b^{i^s} \in X^i$, and since

$$\left(1 - \lambda^{i^s}\right) b^{i^s} + \lambda^{i^s} x^i = a^{i^s}$$

it holds by the convexity of X^i that $a^{i^s} \in X^i$.
Moreover

$$\lambda^{i^s} = \min \left\{ \frac{\varepsilon - \left| w_j^{i^s} - w_j^i \right|}{\varepsilon} \middle| j \in K \right\} \to 1$$

and so

$$\begin{array}{rcll} a_j^{i^s} & = & w_j^{i^s} & \to w_j^i = x_j^i, \quad \forall j \in K, \\ a_j^{i^s} & = & \lambda^{i^s} x_j^i + (1 - \lambda^{i^s})\, w_j^{i^s} & \to x_j^i, \qquad \forall j \in J \cup L. \end{array}$$

Hence there exists an $s^2 \in \mathbb{N}$ such that $s^2 \geq s^1$ and for all $s > s^2$

$$a_j^{i^s} > w_j^{i^s}, \ \forall j \in J, \text{ and } a_j^{i^s} < w_j^{i^s}, \ \forall j \in L.$$

Take $s > s^2$. For all $j \in J$ let

$$\mu_j^{i^s} = \frac{L_j^{i^s}}{a_j^{i^s} - w_j^{i^s}},$$

then $\mu_j^{i^s} \geq 0$ because $a_j^{i^s} - w_j^{i^s} > 0$ and $L_j^{i^s} \geq 0$. For all $j \in L$, let

$$\mu_j^{i^s} = \frac{l_j^{i^s}}{a_j^{i^s} - w_j^{i^s}},$$

then $\mu_j^{i^s} \geq 0$ because $a_j^{i^s} - w_j^{i^s} < 0$ and $l_j^{i^s} \leq 0$. Finally, let

$$\mu_*^{i^s} = \min \left(\left\{ \mu_j^{i^s} \middle| j \in J \cup L \right\} \cup \{1\} \right).$$

Clearly, $0 \leq \mu_*^{i^s} \leq 1$. Next, let

$$c^{i^s} = w^{i^s} + \mu_*^{i^s} \left(a^{i^s} - w^{i^s} \right).$$

Because $a^{i^s}, w^{i^s} \in X^i$ and by the convexity of X^i it holds that $c^{i^s} \in X^i$. Moreover, for $j \in J$

$$\begin{array}{rcl} c_j^{i^s} - w_j^{i^s} & = & \mu_*^{i^s} \left(a_j^{i^s} - w_j^{i^s} \right) \leq \mu_j^{i^s} \left(a_j^{i^s} - w_j^{i^s} \right) = L_j^{i^s}, \\ c_j^{i^s} - w_j^{i^s} & = & \mu_*^{i^s} \left(a_j^{i^s} - w_j^{i^s} \right) \geq 0 \geq l_j^{i^s}, \end{array}$$

for $j \in K$

$$c_j^{i^s} - w_j^{i^s} = \mu_*^{i^s} \left(a_j^{i^s} - w_j^{i^s} \right) = 0 \text{ and so } l_j^{i^s} \leq c_j^{i^s} - w_j^{i^s} \leq L_j^{i^s},$$

and for $j \in L$

$$\begin{array}{rcl} c_j^{i^s} - w_j^{i^s} & = & \mu_*^{i^s} \left(a_j^{i^s} - w_j^{i^s} \right) \geq \mu_j^{i^s} \left(a_j^{i^s} - w_j^{i^s} \right) = l_j^{i^s}, \\ c_j^{i^s} - w_j^{i^s} & = & \mu_*^{i^s} \left(a_j^{i^s} - w_j^{i^s} \right) \leq 0 \leq L_j^{i^s}. \end{array}$$

Further, for $j \in J$

$$\mu_j^{i^s} = \frac{L_j^{i^s}}{a_j^{i^s} - w_j^{i^s}} \to \frac{L_j^i}{x_j^i - w_j^i} \geq \frac{x_j^i - w_j^i}{x_j^i - w_j^i} = 1$$

and for $j \in L$

$$\mu_j^{i^s} = \frac{l_j^{i^s}}{a_j^{i^s} - w_j^{i^s}} \to \frac{l_j^i}{x_j^i - w_j^i} \geq \frac{x_j^i - w_j^i}{x_j^i - w_j^i} = 1.$$

So $\mu_*^{i^s} \to 1$ and therefore

$$c^{i^s} = w^{i^s} + \mu_*^{i^s}\left(a^{i^s} - w^{i^s}\right) \to w^i + x^i - w^i = x^i.$$

Moreover,

$$p^s \cdot \left(c^{i^s} - w^{i^s}\right) \to p \cdot \left(x^i - w^i\right) < 0.$$

Therefore, there exists an $s^3 \in \mathbb{N}$ such that $s^3 \geq s^2$ and $s > s^3$ implies $p^s \cdot (c^{i^s} - w^{i^s}) < 0$. Hence, for $s > s^3$,

$$c^{i^s} \in B^i\left(p^s, l^{i^s}, L^{i^s}, w^{i^s}\right).$$

Construct the sequence $(x^{i^s})_{s \in \mathbb{N}}$ as follows:

$$\begin{array}{lcl} s & \leq & s^3, \quad x^{i^s} \text{ is an arbitrary element of } B^i\left(p^s, l^{i^s}, L^{i^s}, w^{i^s}\right); \\ s & > & s^3, \quad x^{i^s} = c^{i^s}. \end{array}$$

This sequence clearly has all the desired properties.
2.2. $p \cdot (x^i - w^i) = 0$.
Two subsubcases have to be considered.

2.2.1. $l^i = 0$.
2.2.2. $l^i < 0$.

2.2.1. $l^i = 0$.
Since $x^i - w^i \geq l^i$ and $l^i = 0$, $x^i \geq w^i$. Using $p \gg 0$ this implies $x^i = w^i$. Obviously $w^{i^s} \in B^i\left(p^s, l^{i^s}, L^{i^s}, w^{i^s}\right)$. Moreover, $w^{i^s} \to w^i = x^i$. Construct the sequence $(x^{i^s})_{s \in \mathbb{N}}$ as follows,

$$x^{i^s} = w^{i^s}, \text{ for all } s \in \mathbb{N}.$$

This sequence clearly has all the desired properties.
2.2.2. $l^i < 0$.
Define the set

$$J = \left\{ j \,\middle|\, l_j^i < 0 \right\}.$$

Since $w^i \in \text{Int}(X^i)$ there exists an $\alpha > 0$ such that $\|y^i - w^i\|_\infty \leq \alpha$ implies $y^i \in X^i$. There exists an $s^1 \in \mathbb{N}$ such that for all $s > s^1$

$$\left\| w^{i^s} - w^i \right\|_\infty \leq \frac{1}{2}\alpha \text{ and } l_j^{i^s} < 0, \ \forall j \in J.$$

Take $s > s^1$. Define $\varepsilon^s > 0$ by

$$\varepsilon^s = \min\left(\left\{ -l_j^{i^s} \,\middle|\, j \in J \right\} \cup \left\{ \frac{1}{2}\alpha \right\}\right)$$

and define $\underline{w}^{i^s}$ by

$$\begin{aligned}\underline{w}_j^{i^s} &= w_j^{i^s} - \varepsilon^s, \quad j \in J,\\ \underline{w}_j^{i^s} &= w_j^{i^s}, \qquad\quad j \notin J.\end{aligned}$$

The following properties hold for $\underline{w}^{i^s}$:

$$\underline{w}^{i^s} \in X^i,\ l^{i^s} \leq \underline{w}^{i^s} - w^{i^s} \leq 0 \leq L^{i^s}, \text{ and } p^s \cdot \underline{w}^{i^s} < p^s \cdot w^{i^s}. \tag{3}$$

Moreover, $\underline{w}^{i^s} \to \underline{w}^i$ where, for ε equal to $(\min\{-l_j^i \mid j \in J\} \cup \{\frac{1}{2}\alpha\})$,

$$\begin{aligned}\underline{w}_j^{i} &= w_j^{i} - \varepsilon, \quad j \in J,\\ \underline{w}_j^{i} &= w_j^{i}, \qquad\quad j \notin J,\end{aligned}$$

so

$$\underline{w}^i \in X^i,\ l^i \leq \underline{w}^i - w^i \leq 0 \leq L^i, \text{ and } p \cdot \underline{w}^i < p \cdot w^i.$$

Consider the sequence $(c^{i^s})_{s \in \mathbb{N}}$ defined as in Part 2.1 of the proof. It may be assumed that the elements of this sequence satisfy

$$c^{i^s} \in X^i,\ l^{i^s} \leq c^{i^s} - w^{i^s} \leq L^{i^s}, \text{ and } c^{i^s} \to x^i. \tag{4}$$

If $p^s \cdot c^{i^s} > p^s \cdot w^{i^s}$ then define λ^{i^s} by

$$\lambda^{i^s} = \frac{p^s \cdot w^{i^s} - p^s \cdot \underline{w}^{i^s}}{p^s \cdot c^{i^s} - p^s \cdot \underline{w}^{i^s}} \tag{5}$$

and if $p^s \cdot c^{i^s} \leq p^s \cdot w^{i^s}$ then define $\lambda^{i^s} = 1$. Clearly $0 < \lambda^{i^s} \leq 1$. Define

$$d^{i^s} = \underline{w}^{i^s} + \lambda^{i^s}\left(c^{i^s} - \underline{w}^{i^s}\right).$$

Using the convexity of X^i, $d^{i^s} \in X^i$. By (3) and (4)

$$\begin{aligned}d^{i^s} - w^{i^s} &= \lambda^{i^s}\left(c^{i^s} - w^{i^s}\right) + \left(1 - \lambda^{i^s}\right)\left(\underline{w}^{i^s} - w^{i^s}\right) \geq l^{i^s},\\ d^{i^s} - w^{i^s} &= \lambda^{i^s}\left(c^{i^s} - w^{i^s}\right) + \left(1 - \lambda^{i^s}\right)\left(\underline{w}^{i^s} - w^{i^s}\right) \leq L^{i^s}.\end{aligned}$$

If $p^s \cdot c^{i^s} > p^s \cdot w^{i^s}$ then by (5)

$$p^s \cdot d^{i^s} = \left(\frac{p^s \cdot w^{i^s} - p^s \cdot \underline{w}^{i^s}}{p^s \cdot c^{i^s} - p^s \cdot \underline{w}^{i^s}}\right) p^s \cdot c^{i^s} + \left(\frac{p^s \cdot c^{i^s} - p^s \cdot w^{i^s}}{p^s \cdot c^{i^s} - p^s \cdot \underline{w}^{i^s}}\right) p^s \cdot \underline{w}^{i^s} = p^s \cdot w^{i^s},$$

and if $p^s \cdot c^{i^s} \leq p^s \cdot w^{i^s}$ then because $\lambda^{i^s} = 1$

$$p^s \cdot d^{i^s} = p^s \cdot c^{i^s} \leq p^s \cdot w^{i^s}.$$

So $d^{i^s} \in B^i\left(p^s, l^{i^s}, L^{i^s}, w^{i^s}\right)$. Using $c^{i^s} \to x^i$ and $p \cdot x^i = p \cdot w^i$,

$$\frac{p^s \cdot w^{i^s} - p^s \cdot \underline{w}^{i^s}}{p^s \cdot c^{i^s} - p^s \cdot \underline{w}^{i^s}} \to \frac{p \cdot w^i - p \cdot \underline{w}^i}{p \cdot w^i - p \cdot \underline{w}^i} = 1$$

and so $\lambda^{i^s} \to 1$. Consequently

$$d^{i^s} \to \underline{w}^i + \left(x^i - \underline{w}^i\right) = x^i.$$

Construct the sequence $\left(x^{i^s}\right)_{s \in \mathbf{N}}$ as follows:

$$\begin{aligned} s &\leq s^1, \quad x^{i^s} \text{ is an arbitrary element of } B^i\left(p^s, l^{i^s}, L^{i^s}, w^{i^s}\right); \\ s &> s^1, \quad x^{i^s} = d^{i^s}. \end{aligned}$$

This sequence clearly has all the desired properties.

Q.E.D.

From the following lemma it follows that the budget correspondence is convex valued.

Lemma 2.3
For $i \in I_m$, if $X^i \subset \mathbf{R}^n$ and X^i is convex then the set

$$B^i\left(p, l^i, L^i, w^i\right) = \left\{x^i \in X^i \,\middle|\, p \cdot x^i \leq p \cdot w^i,\ l^i \leq x^i - w^i \leq L^i\right\}$$

is convex for all $(p, l^i, L^i, w^i) \in \mathbf{R}^n_{++} \times -\mathbf{R}^n_+ \times \mathbf{R}^n_+ \times Int(X^i)$.

Proof
Take $x^i, y^i \in B^i\left(p, l^i, L^i, w^i\right)$ and $\lambda \in [0,1]$. Then by the convexity of X^i, $\lambda x^i + (1-\lambda) y^i \in X^i$. Moreover,

$$\begin{aligned} & p \cdot (\lambda x^i + (1-\lambda) y^i) \leq \lambda p \cdot w^i + (1-\lambda) p \cdot w^i = p \cdot w^i, \\ & \lambda x^i + (1-\lambda) y^i - w^i = \lambda(x^i - w^i) + (1-\lambda)(y^i - w^i) \geq \lambda l^i + (1-\lambda) l^i = l^i, \\ & \lambda x^i + (1-\lambda) y^i - w^i = \lambda(x^i - w^i) + (1-\lambda)(y^i - w^i) \leq \lambda L^i + (1-\lambda) L^i = L^i. \end{aligned}$$

So $\lambda x^i + (1-\lambda) y^i \in B^i\left(p, l^i, L^i, w^i\right)$.

Q.E.D.

3 Existence of a Continuum of Constrained Equilibria

An equilibrium in an economy with price constraints can be obtained by using rationing to clear the markets. A rationing system describes all rationing schemes permitted in the economy. The rationing system on market $j \in I_n$ is a subset of $-\mathbf{R}^m_+ \times \mathbf{R}^m_+$ specifying all possible ways how excess demand of consumers might be bounded from below and from above. Let Q be the set $\prod_{i=1}^n [0,1]$. It is assumed that the rationing system can be described by the functions $\tilde{l} : Q \times \prod_{i=1}^m \text{Int}\left(X^i\right) \to -\mathbf{R}^{mn}_+$ and $\tilde{L} : Q \times \prod_{i=1}^m \text{Int}\left(X^i\right) \to \mathbf{R}^{mn}_+$. The component $(i-1)n + j$ of $\tilde{l}$ is denoted by $\tilde{l}^i_j$ and the component $(i-1)n + j$ of $\tilde{L}$ by $\tilde{L}^i_j$, $\forall i \in I_m$, $\forall j \in I_n$. Let the initial endowments w be given. Then for $q^1 \in Q$, $\tilde{l}^i_j\left(q^1, w\right)$ is equal to the maximal excess supply of consumer $i \in I_m$ on market $j \in I_n$, and for $q^2 \in Q$,

$\tilde{L}^i_j(q,w)$ is equal to the maximal excess demand of consumer $i \in I_m$ on market $j \in I_n$. It is assumed that for every $i \in I_m$ the functions $\tilde{l}^i_j(\cdot,w)$ and $\tilde{L}^i_j(\cdot,w)$ only depend on q^1_j and q^2_j, respectively. Moreover, the following assumption is made, which guarantees that the rationing system is flexible enough to prove the existence of a constrained equilibrium.

A5. Let the allocation of initial endowments $w \in \prod_{i=1}^m \mathrm{Int}\,(X^i)$ be given. Then the functions $\tilde{l}\,(\cdot,w)$ and $\tilde{L}\,(\cdot,w)$ are continuous on Q and satisfy for all $i \in I_m$, $j \in I_n$, $q^1 \in Q$,

$$\tilde{l}^i_j\left(q^1,\ w\right) = 0, \text{ if } q^1_j = 0,$$
$$\tilde{l}^i_j\left(q^1,\ w\right) < -w^i_j, \text{ if } q^1_j = 1,$$

and for all $i \in I_m$, $j \in I_n$, $q^2 \in Q$,

$$\tilde{L}^i_j\left(q^2,\ w\right) > \sum_{h\neq i} w^h_j, \text{ if } q^2_j = 0,$$
$$\tilde{L}^i_j\left(q^2,\ w\right) = 0, \text{ if } q^2_j = 1.$$

Below a few examples are given to illustrate the generality of the way the rationing system is described. Only the expressions for the rationing of the excess supply are given. Similar expressions for the rationing of the excess demand are easily obtained. It is not difficult to verify that the examples satisfy Assumption A5.

Uniform rationing as described in Drèze (1975) is obtained by specifying

$$\tilde{l}^i_j(q^1,w) = -q^1_j \sum_{h=1}^m w^h_j.$$

Let a real number $\varepsilon > 0$ be given. Then rationing determined by initial endowments, equivalently to the rationing system in Kurz (1982), is obtained by specifying

$$\tilde{l}^i_j(q^1,w) = -(1+\varepsilon)q^1_j w^i_j.$$

Rationing determined by market share as in Weddepohl (1983), let numbers $\alpha^i_j > 0$ be given such that $\sum_{i=1}^m \alpha^i_j = 1$. Then,

$$\tilde{l}^i_j(q^1,w) = -\frac{\alpha^i_j q^1_j \sum_{h=1}^m w^h_j}{\min\left\{\alpha^h_j \mid h \in I_m\right\}}.$$

Rationing determined by priority (for a special case see Weddepohl (1987)), let $\sigma_j : I_m \to I_m$ be a permutation specifying the order in which consumers are rationed on their excess supply on market j. Then,

$$\tilde{l}^i_j(q^1,w) = -\max\left\{\sigma_j(i) - m + mq^1_j, 0\right\} \sum_{h=1}^m w^h_j.$$

It is even possible to allow for the rationing system consisting of all kinds of rationing schemes. So every element in the set $-\prod_{i=1}^{m}\prod_{j=1}^{n}[0,\sum_{h=1}^{m} w_j^h]$ is allowed as a rationing scheme on excess supply and every element in the set $\prod_{i=1}^{m}\prod_{j=1}^{n}[0,\sum_{h=1}^{m} w_j^h]$ is allowed as a rationing scheme on excess demand. To construct the functions $\tilde{l}$ and $\tilde{L}$ the theorem of Peano about the existence of a space-filling curve is used. For an easy treatise of this theorem, see Section 2.3 of Armstrong (1983). There a continuous function f^1 is constructed from $[0,1]$ onto a triangle. It is not difficult to construct a continuous function f^2 from the triangle onto $\prod_{k=1}^{2}[0,1]$. Then the function $f^2 \circ f^1$ is a continuous function from $[0,1]$ onto $\prod_{k=1}^{2}[0,1]$. Using the function $f^2 \circ f^1$ it is not difficult to construct a continuous function g^2 from $[0,1]$ onto $\prod_{k=1}^{2}[0,1]$ having the additional property that $g^2(0) = (0,0)$ and $g^2(1) = (1,1)$. For $K \geq 3$ define $g^K : \prod_{k=1}^{K-1}[0,1] \to \prod_{k=1}^{K}[0,1]$ by

$$g^K(x_1,\ldots,x_{K-1}) = \left(g^{K-1}(x_1,\ldots,x_{K-2}), x_{K-1}\right), \ \forall x \in \prod_{k=1}^{K-1}[0,1].$$

Clearly g^K is continuous and onto for all $K \geq 2$. Let the function $h^2 : [0,1] \to \prod_{k=1}^{2}[0,1]$ be equal to g^2. For $K \geq 3$ define $h^K : [0,1] \to \prod_{k=1}^{K}[0,1]$ by

$$h^K(x) = g^K\left(h^{K-1}(x)\right), \ x \in [0,1].$$

Then h^K is continuous and onto for all $K \geq 2$. Now a rationing system where all possible kinds of rationing schemes are allowed can be obtained by taking

$$\tilde{l}_j^i(q^1,w) = -h_i^m(q_j^1)\sum_{h=1}^{m} w_j^h$$

and taking $\tilde{L}_j^i(q^2,w)$ similarly. It is easily verified that by the way the function h^m is constructed, it is guaranteed that Assumption A5 is satisfied and that all possible rationing schemes are obtained. It is possible to construct the rationing system of the last example because, contrary to other papers, it is not assumed that $\tilde{l}_j^i$ or $\tilde{L}_j^i$ are monotone functions.

Definition 3.1 (Constrained Equilibrium Satisfying a Rationing System)
A constrained equilibrium of the economy $E = (\{X^i, \succeq^i, w^i\}_{i=1}^m, P_{(\underline{p},\bar{p})})$ with rationing system $(\tilde{l},\tilde{L})$ is an element $(x^{*1},\ldots,x^{*m},l^{*1},\ldots,l^{*m},L^{*1},\ldots,L^{*m},p^*)$ of the set $\prod_{i=1}^{m} X^i \times \prod_{i=1}^{m} -\mathbf{R}_+^n \times \prod_{i=1}^{m}\mathbf{R}_+^n \times P_{(\underline{p},\bar{p})}$ such that the four conditions of Definition 2.1 are satisfied and, additionally, there exists a $q^{*1} \in Q$ and a $q^{*2} \in Q$ such that

$$\begin{aligned} \tilde{l}(q^{*1},w) &= (l^{*1},\ldots,l^{*m}), \\ \tilde{L}(q^{*2},w) &= (L^{*1},\ldots,L^{*m}). \end{aligned}$$

Let R be the subset of $\mathbf{R}_{++}^{2n}$ given by

$$R = \left\{r \in \mathbf{R}_{++}^{2n} \mid r_j \leq r_{j+n}, \ \forall j \in I_n\right\}.$$

Notice that an element $r \in R$ can be used to specify a set of admissible prices P_r satisfying Assumption A4. Define the components $j \in I_n$ of the function $\hat{p} : Q \times R \to \mathbf{R}^n_{++}$ by

$$\hat{p}_j(q, r) = \max\{r_j, \min\{r_j(2 - 3q_j) + r_{j+n}(3q_j - 1), r_{j+n}\}\}, \ \forall (q, r) \in Q \times R.$$

Let the functions $\tilde{l}$ and $\tilde{L}$ specifying a rationing system be given. Then define the functions $\hat{l} : Q \times \prod_{i=1}^m \mathrm{Int}(X^i) \to -\mathbf{R}^{mn}_+$ and $\hat{L} : Q \times \prod_{i=1}^m \mathrm{Int}(X^i) \to \mathbf{R}^{mn}_+$ by

$$\begin{aligned} \hat{l}(q, w) &= \tilde{l}(\min\{\iota^n, 3q\}, w), \ \forall (q, w) \in Q \times \prod_{i=1}^m \mathrm{Int}\left(X^i\right), \\ \hat{L}(q, w) &= \tilde{L}(\max\{0, 3q - 2\iota^n\}, w), \ \forall (q, w) \in Q \times \prod_{i=1}^m \mathrm{Int}\left(X^i\right), \end{aligned}$$

where ι^n denotes the n-dimensional vector containing only ones. The minimum and maximum are taken componentswise. These functions are used in the equilibrium existence proofs. Notice that if $q_j \geq \frac{1}{3}$ then $\hat{l}^i_j(q, w) < -w^i_j$ and if $q_j \leq \frac{2}{3}$ then $\hat{L}^i_j(q, w) > \sum_{h \neq i} w^i_j$.

Theorem 3.2
Let be given the economy $E = (\{X^i, \succeq^i, w^i\}_{i=1}^m, P_{(\underline{p}, \bar{p})})$ with rationing system $(\tilde{l}, \tilde{L})$ and let the Assumptions A1, A2, A3, A4, and A5 be satisfied. If $q^* \in Q$ is such that there exists

$$x^{*i} \in \delta^i\left(\hat{p}(q^*, \underline{p}, \bar{p}), \hat{l}^i(q^*, w), \hat{L}^i(q^*, w), w^i\right) \tag{6}$$

for all $i \in I_m$ such that

$$\sum_{i=1}^m x^{*i} = \sum_{i=1}^m w^i, \tag{7}$$

then q^* induces a constrained equilibrium $(x^{*1}, \ldots, x^{*m}, \hat{l}(q^*, w), \hat{L}(q^*, w), \hat{p}(q^*, \underline{p}, \bar{p}))$ of the economy E with rationing system $(\tilde{l}, \tilde{L})$.

Proof
By the definitions of $\hat{p}$, $\hat{l}$, and $\hat{L}$, $\hat{p}(q^*, \underline{p}, \bar{p}) \in P_{(\underline{p}, \bar{p})}$, and there are $q^{*1} \in Q$ and $q^{*2} \in Q$ such that $\hat{l}(q^*, w) = \tilde{l}(q^{*1}, w) \leq 0$ and $\hat{L}(q^*, w) = \tilde{L}(q^{*2}, w) \geq 0$. Therefore the additional conditions of Definition 3.1 are satisfied. Conditions 1 and 2 of Definition 2.1 of a constrained equilibrium, requiring that

$$x^{*i} \in \delta^i\left(\hat{p}(q^*, \underline{p}, p), \hat{l}^i(q^*, w), \hat{L}^i(q^*, w), w^i\right),$$

and

$$\sum_{i=1}^m x^{*i} = \sum_{i=1}^m w^i,$$

respectively, are satisfied, using (6) and (7). Now the Conditions 3 and 4 of Definition 2.1 are examined. Clearly, for all $i \in I_m$ and $j \in I_n$,

$$x^{*i}_j - w^i_j = \sum_{h=1}^m w^h_j - \sum_{h \neq i} x^{*h}_j - w^i_j \leq \sum_{h \neq i} w^h_j \tag{8}$$

and

$$x_j^{*i} - w_j^i \geq -w_j^i. \tag{9}$$

If for some $k \in I_n$ there is an $h \in I_m$ such that $x_k^{*h} - w_k^h = \hat{L}_k^h(q^*, w)$ then by (8) $\hat{L}_k^h(q^*, w) \leq \sum_{i \neq h} w_k^i$ and therefore $q_k^* > \frac{2}{3}$ and hence $\hat{l}_k^i(q^*, w) < -w_k^i$, $\forall i \in I_m$. Then (9) implies

$$x_k^{*i} - w_k^i > \hat{l}_k^i(q^*, w), \ \forall i \in I_m,$$

thereby satisfying the first part of Condition 3.
If for some $k \in I_n$ there is an $h \in I_m$ such that $x_k^{*h} - w_k^h = \hat{l}_k^h(q^*, w)$ then by (9) $\hat{l}_k^h(q^*, w) \geq -w_k^h$ and therefore $q_k^* < \frac{1}{3}$ and hence $\hat{L}_k^i(q^*, w) > \sum_{h \neq i} w_k^h$, $\forall i \in I_m$. Then (8) implies

$$x_k^{*i} - w_k^i < \hat{L}_k^i(q^*, w), \ \forall i \in I_m,$$

thereby satisfying the second part of Condition 3.
If for some $k \in I_n$, $\hat{p}_k(q^*, \underline{p}, \bar{p}) < \bar{p}_k$ then $q_k^* < \frac{2}{3}$ and therefore $\hat{L}_k^i(q^*, w) > \sum_{h \neq i} w_k^h$, $\forall i \in I_m$, and this implies by (8) that

$$x_k^{*i} - w_k^i < \hat{L}_k^i(q^*, w), \ \forall i \in I_m.$$

Also if for some $k \in I_n$, $\hat{p}_k(q^*, \underline{p}, \bar{p}) > \underline{p}_k$ then $q_k^* > \frac{1}{3}$ and therefore $\hat{l}_k^i(q^*, w) < -w_k^i$, $\forall i \in I_m$, and this implies by (9) that

$$x_k^{*i} - w_k^i > \hat{l}_k^i(q^*, w), \ \forall i \in I_m.$$

So Condition 4 is also satisfied.

Q.E.D.

The following theorem shows that there is no loss of generality if the functions $\hat{l}, \hat{L}$ and $\hat{p}$ are used, in the sense that all constrained equilibrium allocations are obtained.

Theorem 3.3

Let be given the economy $E = (\{X^i, \succeq^i, w^i\}_{i=1}^m, P_{(\underline{p}, \bar{p})})$ with rationing system $(\tilde{l}, \tilde{L})$ and let the Assumptions A1, A2, A3, A4, and A5 be satisfied. If $q^{*1} \in Q$ and $q^{*2} \in Q$ is such that $(x^{*1}, \ldots, x^{*m}, \tilde{l}(q^{*1}, w), \tilde{L}(q^{*2}, w), p^*)$ is a constrained equilibrium of the economy E with rationing system $(\tilde{l}, \tilde{L})$, then there exists a $q^* \in Q$ such that

$$(x^{*1}, \ldots, x^{*m}, \hat{l}(q^*, w), \hat{L}(q^*, w), \hat{p}(q^*, \underline{p}, \bar{p}))$$

is a constrained equilibrium of the economy E with rationing system $(\tilde{l}, \tilde{L})$.

Proof

Define

$$\begin{aligned} J &= \left\{ j \in I_n \mid \exists i \in I_m \text{ such that } x_j^{*i} - w_j^i = \tilde{l}_j^i(q^{*1}, w) \right\}, \\ K &= \left\{ j \in I_n \mid \exists i \in I_m \text{ such that } x_j^{*i} - w_j^i = \tilde{L}_j^i(q^{*2}, w) \right\}, \\ L &= I_n \setminus (J \cup K). \end{aligned}$$

By Condition 3 of Definition 2.1 $\{J, K, L\}$ is a partition of I_n. By Condition 4 of Definition 2.1, $p_j^* = \underline{p}_j$, $\forall j \in J$, and $p_j^* = \bar{p}_j$, $\forall j \in K$. Moreover $\underline{p}_j \leq p_j^* < \bar{p}_j$, $\forall j \in L$. Define q^* by

$$
\begin{aligned}
q_j^* &= \tfrac{1}{3} q_j^{*1}, && \forall j \in J, \\
q_j^* &= \tfrac{2}{3} + \tfrac{1}{3} q_j^{*2}, && \forall j \in K, \\
q_j^* &= \tfrac{1}{2}, && \forall j \in L \text{ with } \underline{p}_j = \bar{p}_j, \\
q_j^* &= \tfrac{p_j^* + \bar{p}_j - 2\underline{p}_j}{3(\bar{p}_j - \underline{p}_j)}, && \forall j \in L \text{ with } \underline{p}_j < \bar{p}_j.
\end{aligned}
$$

Clearly, for all $i \in I_m$ it holds that

$$\hat{l}_j^i(q^*, w) = \tilde{l}_j^i(q^{*1}, w), \forall j \in J, \tag{10}$$

$$\hat{L}_j^i(q^*, w) = \tilde{L}_j^i(q^{*2}, w), \forall j \in K, \tag{11}$$

$$\hat{p}(q^*, \underline{p}, \bar{p}) = p^*, \forall j \in I_n. \tag{12}$$

By Condition 3 of Definition 2.1 and the definition of L it holds for all $i \in I_m$ that

$$\forall j \in J \cup L : x_j^{*i} - w_j^i < \tilde{L}_j^i(q^{*2}, w), \tag{13}$$

$$\forall j \in K \cup L : \tilde{l}_j^i(q^{*1}, w) < x_j^{*i} - w_j^i. \tag{14}$$

Moreover, for all $i \in I_m$,

$$\forall j \in J \cup L : \hat{L}_j^i(q^*, w) > \sum_{h \neq i} w_j^h, \tag{15}$$

$$\forall j \in K \cup L : \hat{l}_j^i(q^*, w) < -w_j^i. \tag{16}$$

If it is shown that for all $i \in I_m$

$$x^{*i} \in \delta^i\left(\hat{p}(q^*, \underline{p}, \bar{p}), \hat{l}^i(q^*, w), \hat{L}^i(q^*, w), w^i\right),$$

then Theorem 3.3 is proved, using Theorem 3.2. Suppose that for some $h \in I_m$, $x^{*h} \notin \delta^h(\hat{p}(q^*, \underline{p}, \bar{p}), \hat{l}^h(q^*, w), \hat{L}^h(q^*, w), w^h)$. Clearly $x^{*h} \in B^h(\hat{p}(q^*, \underline{p}, \bar{p}), \hat{l}^h(q^*, w), \hat{L}^h(q^*, w), w^h)$ by (10), (11), (12), (15), and (16). Hence for some

$$y^h \in B^h(\hat{p}(q^*, \underline{p}, \bar{p}), \hat{l}^h(q^*, w), \hat{L}^h(q^*, w), w^h)$$

it holds that

$$u^h(y^h) > u^h(x^{*h}) \tag{17}$$

and $y^h \notin B^h(\hat{p}(q^*, \underline{p}, \bar{p}), \tilde{l}^h(q^{*1}, w), \tilde{L}^h(q^{*2}, w), w^h)$. Define for all $\lambda \in (0, 1)$, $y^h(\lambda)$ by

$$y^h(\lambda) = \lambda y^h + (1 - \lambda) x^{*h}.$$

Using Lemma 2.3 it is clear that

$$y^h(\lambda) \in B^h\left(\hat{p}(q^*, \underline{p}, \bar{p}), \hat{l}^h(q^*, w), \hat{L}^h(q^*, w), w^h\right). \tag{18}$$

Using the convexity of $\succeq^h$ and (17) gives $u^h(y^h(\lambda)) > u^h(x^{*h})$. However, if λ is sufficiently close to zero, then $y^h(\lambda) \in B^h(\hat{p}(q^*, \underline{p}, \bar{p}), \tilde{l}^h(q^{*1}, w), \tilde{L}^h(q^{*2}, w), w^h)$ using (10), (11), (13), (14) and (18). This gives a contradiction, because

$$x^{*h} \in \delta^h(\hat{p}(q^*, \underline{p}, p), \tilde{l}^h(q^{*1}, w), \tilde{L}^h(q^{*2}, w), w^h).$$

Q.E.D.

The following theorem makes clear that there exists a continuum of constrained equilibria. Using somewhat stronger assumptions than Assumptions A1 and A2, a similar result is obtained in van der Laan and Talman (1990) for the case of the uniform rationing system.

Theorem 3.4

Let be given the economy $E = (\{X^i, \succeq^i, w^i\}_{i=1}^m, P_{(\underline{p},\bar{p})})$ with rationing system $(\tilde{l}, \tilde{L})$ and let the Assumptions A1, A2, A3, A4, and A5 be satisfied. Let $k \in I_n$ and $\lambda \in [0,1]$ be given. Then there exists a $q^* \in Q$ such that $q_k^* = \lambda$ and q^* induces a constrained equilibrium

$$\left(x^{*1}, \ldots, x^{*m}, \hat{l}\,(q^*, w)\,, \hat{L}\,(q^*, w)\,, \hat{p}(q^*, \underline{p}, \bar{p})\right)$$

of the economy E with rationing system $(\tilde{l}, \tilde{L})$.

Proof

For all $i \in I_m$, define the set $\bar{X}^i$ by

$$X^i = \left\{ x^i \in X^i \left| x_j^i \leq \frac{\bar{p} \cdot w^i + \left(\underline{p}_j - \bar{p}_j\right) w_j^i}{\underline{p}_j}, \ \forall j \in I_n \right. \right\},$$

and define $Q_{k,\lambda} = \{q \in Q \,|q_k = \lambda\}$. Clearly, $Q_{k,\lambda}$ and all $\bar{X}^i$ are compact, convex and non-empty sets in $\mathbf{R}^n$. Define the correspondence $\mu_{k,\lambda} : \prod_{i=1}^m \bar{X}^i \to Q_{k,\lambda}$ by

$$\mu_{k,\lambda}\left(x^1, \ldots, x^m\right) = \left\{ q \in Q_{k,\lambda} \left| q \cdot \sum_{i=1}^m \left(x^i - w^i\right) \geq \right. \right.$$

$$\left. \bar{q} \cdot \sum_{i=1}^m \left(x^i - w^i\right), \ \forall \bar{q} \in Q_{k,\lambda} \right\}, \ \forall \left(x^1, \ldots, x^m\right) \in \prod_{i=1}^m \bar{X}^i.$$

Obviously, if $j \in I_n$ and $j \neq k$ then

$$\sum_{i=1}^m \left(x_j^i - w_j^i\right) > 0 \text{ and } q \in \mu_{k,\lambda}\left(x^1, \ldots, x^m\right) \Rightarrow q_j = 1, \tag{19}$$

$$\sum_{i=1}^m \left(x_j^i - w_j^i\right) < 0 \text{ and } q \in \mu_{k,\lambda}\left(x^1, \ldots, x^m\right) \Rightarrow q_j = 0. \tag{20}$$

Notice that for all $i \in I_m$ and $q \in Q$, $x^i \in \delta^i(\hat{p}(q, \underline{p}, \bar{p}), \hat{l}^i(q, w), \hat{L}^i(q, w), w^i)$ implies $x^i \in \bar{X}^i$. For all $i \in I_m$ define the correspondence $\bar{\delta}^i : Q_{k,\lambda} \to \bar{X}^i$ by $\bar{\delta}^i\,(q) = \delta^i(\hat{p}(q, \underline{p}, \bar{p}), \hat{l}^i(q, w), \hat{L}^i(q, w), w^i)$, for all $q \in Q_{k,\lambda}$, and the correspondence $\varphi : \prod_{i=1}^{m} \bar{X}^i \times Q_{k,\lambda} \to \prod_{i=1}^{m} \bar{X}^i \times Q_{k,\lambda}$ by

$$\varphi\left(x^1, \ldots, x^m, q\right) = \prod_{i=1}^{m} \bar{\delta}^i(q) \times \mu_{k,\lambda}\left(x^1, \ldots, x^m\right), \ \forall \left(x^1, \ldots, x^m, q\right) \in \prod_{i=1}^{m} \bar{X}^i \times Q_{k,\lambda}.$$

Recalling that $Q_{k,\lambda}$ is non-empty and compact it follows easily from the maximum theorem that $\mu_{k,\lambda} : \prod_{i=1}^{m} \bar{X}^i \to Q_{k,\lambda}$ is non-empty valued and upper semi-continuous on $\prod_{i=1}^{m} \bar{X}^i$. By Theorem 2.2, the continuity of $\tilde{l}(\cdot, w), \tilde{L}(\cdot, w)$ and $\hat{p}(\cdot, \underline{p}, \bar{p})$, the continuity of the functions f and g defined by $f(x) = \min\{\iota^n, 3x\}$ and $g(x) = \max\{0, 3x - 2\iota^n\}$, $x \in Q$, the continuity of the utility functions, and by the maximum theorem, it follows that the correspondence $\delta^i(\hat{p}(\cdot, \underline{p}, \bar{p}), \hat{l}^i(\cdot, w), \hat{L}^i(\cdot, w), w^i)$ is non-empty valued and upper semi-continuous on Q. Therefore the correspondence $\bar{\delta}^i$ is non-empty valued and upper semi-continuous on $Q_{k,\lambda}$. Hence the correspondence φ is non-empty valued and upper semi-continuous on $\prod_{i=1}^{m} \bar{X}^i \times Q_{k,\lambda}$. Let $(x^1, \ldots, x^m, q) \in \prod_{i=1}^{m} \bar{X}^i \times Q_{k,\lambda}$. Then the set

$$\prod_{i=1}^{m} \bar{\delta}^i(q) \times \mu_{k,\lambda}\left(x^1, \ldots, x^m\right)$$

is convex, because for all $i \in I_m$ and $q \in Q_{k,\lambda}$, $\bar{\delta}^i(q)$ is convex by the convexity of $\succeq^i$ and the convexity of the budget correspondence (Lemma 2.3), and because $\mu_{k,\lambda}(x^1, \ldots, x^m)$ is convex for all $(x^1, \ldots, x^m) \in \prod_{i=1}^{m} \bar{X}^i$ as is easily verified. Obviously, the set $\prod_{i=1}^{m} \bar{X}^i \times Q_{k,\lambda}$ is convex, compact, and non-empty. Hence all conditions of Kakutani's fixed point theorem are satisfied and the correspondence φ has a fixed point

$$\left(x^{*1}, \ldots, x^{*m}, q^*\right) \in \prod_{i=1}^{m} \bar{X}^i \times Q_{k,\lambda}$$

satisfying

$$x^{*i} \in \bar{\delta}^i(q^*) = \delta^i\left(\hat{p}(q^*, \underline{p}, \bar{p}), \hat{l}^i(q^*, w), \hat{L}^i(q^*, w), w^i\right), \ \forall i \in I_m, \tag{21}$$

and

$$q^* \in \mu_{k,\lambda}\left(x^{*1}, \ldots, x^{*m}\right).$$

It will be shown that all the conditions of a constrained equilibrium are satisfied at

$$\left(x^{*1}, \ldots, x^{*m}, \hat{l}(q^*, w), \hat{L}(q^*, w), \hat{p}(q^*, \underline{p}, \bar{p})\right).$$

Condition 1 follows immediately by (21). Using Assumptions A1 and A2 it is clear that

$$\hat{p}\left(q^*, \underline{p}, \bar{p}\right) \cdot \sum_{i=1}^{m}\left(x^{*i} - w^i\right) = 0. \tag{22}$$

Suppose there exists an $h \in I_n$ such that

$$\sum_{i=1}^{m} x_h^{*i} < \sum_{i=1}^{m} w_h^i. \tag{23}$$

Two cases are possible, $h \neq k$ and $h = k$.

If $h \neq k$ then by the definition of $\mu_{k,\lambda}$, $q_h^* = 0$, and so for all $i \in I_m$, $\hat{l}_h^i(q^*, w) = 0$. Consequently for all $i \in I_m$, $x_h^{*i} \geq w_h^i$, and therefore $\sum_{i=1}^m x_h^{*i} \geq \sum_{i=1}^m w_h^i$, a contradiction with (23).

Suppose $h = k$. Then clearly $\hat{p}_k(q^*, \underline{p}, p) \geq \underline{p}_k > 0$. This implies using (22) that there is a $g \in I_n$, $g \neq k$, such that

$$\sum_{i=1}^{m} x_g^{*i} > \sum_{i=1}^{m} w_g^i. \tag{24}$$

This implies using the definition of $\mu_{k,\lambda}$, that $q_g^* = 1$ and so $\hat{L}_g^i(q^*, w) = 0$, for all $i \in I_m$. Hence, for all $i \in I_m$, $x_g^{*i} \leq w_g^i$ and therefore $\sum_{i=1}^m x_g^{*i} \leq \sum_{i=1}^m w_g^i$. This is a contradiction with (24). Consequently

$$\sum_{i=1}^{m} x^{*i} \geq \sum_{i=1}^{m} w^i. \tag{25}$$

From (22), (25), and $\hat{p}\left(q^*, \underline{p}, \bar{p}\right) \gg 0$, it follows that $\sum_{i=1}^m x^{*i} = \sum_{i=1}^m w^i$. Hence Condition 2 of a constrained equilibrium is satisfied. Conditions 3, 4 and the conditions imposed by the rationing system $(\tilde{l}, \tilde{L})$ follow from Theorem 3.2.

Q.E.D.

Notice that if $\lambda \in [0, \frac{1}{3})$ it cannot be guaranteed that for all $i \in I_m$ there exists some $j \in I_n$ such that $\hat{l}_j^i(q^*) < 0$. Therefore the original proof of Drèze (1975) of the continuity of the budget correspondence of consumer $i \in I_m$ is not sufficient to prove the existence of constrained equilibria corresponding with $\lambda \in [0, \frac{1}{3})$.

Suppose that one of the commodities is a numeraire commodity. A constrained equilibrium without rationing of the numeraire commodity is also called a Drèze equilibrium. It is easy to prove the existence of such a Drèze equilibrium with the aid of Theorem 3.4.

Corollary 3.5

Let one of the commodities be a numeraire commodity. Then there exists a Drèze equilibrium of the economy $E = (\{X^i, \succeq^i, w^i\}_{i=1}^m, P_{(\underline{p},\bar{p})})$ with rationing system $(\tilde{l}, \tilde{L})$ if Assumptions A1, A2, A3, A4, and A5 are satisfied.

Proof

Suppose commodity $k \in I_n$ is the numeraire commodity. Take $\lambda = \frac{1}{2}$ and apply Theorem 3.4. Then there exists a constrained equilibrium induced by some $q^* \in Q_{k,\frac{1}{2}}$. Clearly, for all $i \in I_m$, $\hat{l}_k^i(q^*, w) < -w_k^i$ and $\hat{L}_k^i(q^*, w) > \sum_{h \neq i} w_k^h$, and consequently the numeraire commodity is not rationed.

Q.E.D.

4 Existence of Extended Supply Constrained Equilibria

In van der Laan (1980) and Kurz (1982) it has been remarked that in real world economies supply rationing occurs more frequently than demand rationing. Examples are the rationing of labour supplied, resulting in unemployment, and quotas on the supply of agricultural products. A supply constrained equilibrium is a constrained equilibrium without binding rationing on the excess demand, while on one market there is no binding rationing at all. The commodity of this last market can be chosen ex post as an unrationed numeraire commodity. It is possible to give a generalization of the concept of a supply constrained equilibrium. Let a vector $\alpha \in Q$ be given. A constrained equilibrium where for each market $j \in I_n$ the equilibrium pseudo price is less than or equal to α_j while on at least one market $k \in I_n$ the pseudo-price is equal to α_k is called an extended supply constrained equilibrium with respect to α. In this way it can be modelled for example that on some markets a given amount of rationing on excess demand is allowed, or that on some markets some rationing on excess supply is present.

Define the n-dimensional unit simplex by $S^n = \{q \in \mathbf{R}^n_+ \mid \sum_{j=1}^n q_j = 1\}$. Obviously, S^n is non-empty, convex, and compact. For $i \in I_m$, let $w^i \in \text{Int}(X^i) \subset \mathbf{R}^n_+$ be given. Moreover, let $\underline{p}, \bar{p} \in \mathbf{R}^n_{++}$ satisfying $\underline{p} \leq \bar{p}$ be given. Then define $\bar{X}^i$, $i \in I_n$, as in the proof of Theorem 3.4. Define the correspondence μ from $\prod_{i=1}^m \bar{X}^i$ into S^n by

$$\mu\left(x^1, \ldots, x^m\right) = \left\{ q \in S^n \,\middle|\, q \cdot \sum_{i=1}^m \left(x^i - w^i\right) \geq q \cdot \sum_{i=1}^m \left(x^i - w^i\right),\ \forall q \in S^n \right\},\ \forall \left(x^1, \ldots, x^m\right) \in \prod_{i=1}^m X^i.$$

Lemma 4.1

For $i \in I_m$, let $w^i \in Int(X^i) \subset \mathbf{R}^n_+$ and let $\underline{p}, \bar{p} \in \mathbf{R}^n_{++}$ satisfying $\underline{p} \leq \bar{p}$ be given. Then the correspondence μ from $\prod_{i=1}^m X^i$ into S^n is upper semi-continuous and $\mu\left(x^1, \ldots, x^m\right)$ is non-empty and convex for all $(x^1, \ldots, x^m) \in \prod_{i=1}^m \bar{X}^i$. Further, if $(x^1, \ldots, x^m) \in \prod_{i=1}^m \bar{X}^i$ and for some $g, h \in I_n$ it holds that $\sum_{i=1}^m x^i_g - \sum_{i=1}^m w^i_g > \sum_{i=1}^m x^i_h - \sum_{i=1}^m w^i_h$, then $q_h = 0$ for all $q \in \mu\left(x^1, \ldots, x^m\right)$.

Proof

The non-empty valuedness and the upper semi-continuity of μ follow immediately using the maximum theorem. That μ is convex valued is also easily verified. Finally, let $g, h \in I_n$ be such that

$$\sum_{i=1}^m \left(x^i_g - w^i_g\right) > \sum_{i=1}^m \left(x^i_h - w^i_h\right).$$

Suppose $q_h > 0$, then take $\bar{q} \in S^n$ as follows:

$$\begin{aligned} \bar{q}_j &= q_j, && j \neq g, h, \\ q_g &= q_g + q_h, && \\ \bar{q}_h &= 0. && \end{aligned}$$

Hence,

$$\bar{q}\cdot\sum_{i=1}^{m}\left(x^i-w^i\right)=q\cdot\sum_{i=1}^{m}\left(x^i-w^i\right)+q_h\left[\sum_{i=1}^{m}\left(x_g^i-w_g^i\right)-\sum_{i=1}^{m}\left(x_h^i-w_h^i\right)\right]>q\cdot\sum_{i=1}^{m}\left(x^i-w^i\right),$$

which is a contradiction.

Q.E.D.

The existence of an extended supply constrained equilibria with respect to an arbitrary $\alpha \in Q$ is a consequence of the following theorem.

Theorem 4.2

Let be given the economy $E = (\{X^i, \succeq^i, w^i\}_{i=1}^m, P_{(\underline{p},\bar{p})})$ with rationing system $(\tilde{l}, \tilde{L})$ and let the Assumptions A1, A2, A3, A4, and A5 be satisfied. Then for every $\alpha \in Q$ there exists an extended supply constrained equilibrium with respect to α,

$$\left(x^{*1},\ldots,x^{*m},\hat{l}\left(q^*,w\right),\hat{L}\left(q^*,w\right),\hat{p}(q^*,\underline{p},\bar{p})\right),$$

of the economy E with rationing system $(\tilde{l}, \tilde{L})$, i.e. $q^* \in Q$ is such that $q^* \leq \alpha$ and for some $k \in I_n$, $q_k^* = \alpha_k$.

Proof

Define the component $j \in I_n$ of the function $f : S^n \to Q$ by

$$f_j\left(\bar{q}\right)=\frac{\alpha_j\bar{q}_j}{\max\left\{\bar{q}_1,\ldots,\bar{q}_n\right\}},\ \forall\bar{q}\in S^n.$$

Clearly f is a continuous function on S^n. Define the correspondence $\tilde{\delta}^i : S^n \to \bar{X}^i$ by $\tilde{\delta}^i(\bar{q}) = \bar{\delta}^i(f(\bar{q}))$, where $\bar{\delta}^i$ is as defined in the proof of Theorem 3.4. Since f is continuous on S^n and $\bar{\delta}^i$ is upper semi-continuous on Q it follows that $\tilde{\delta}^i$ is upper semi-continuous on S^n. Define the correspondence $\varphi : \prod_{i=1}^m \bar{X}^i \times S^n \to \prod_{i=1}^m \bar{X}^i \times S^n$ by

$$\varphi\left(x^1,\ldots,x^m,\bar{q}\right)=\prod_{i=1}^{m}\tilde{\delta}^i\left(\bar{q}\right)\times\mu\left(x^1,\ldots,x^m\right),\ \forall\left(x^1,\ldots,x^m,\bar{q}\right)\in\prod_{i=1}^{m}\bar{X}^i\times S^n.$$

This correspondence is upper semi-continuous on $\prod_{i=1}^m \bar{X}^i \times S^n$ by Lemma 4.1 and the upper semi-continuity of $\tilde{\delta}^i$, $\forall i \in I_m$. For all $(x^1,\ldots,x^m,\bar{q}) \in \prod_{i=1}^m \bar{X}^i \times S^n$, the convexity of the set $\prod_{i=1}^m \tilde{\delta}^i(\bar{q}) \times \mu(x^1,\ldots,x^m)$ is guaranteed by Lemma 4.1 and the remarks made in the proof of Theorem 3.4 with respect to the convexity of the set $\bar{\delta}^i(\bar{q})$. The set $\prod_{i=1}^m \bar{X}^i \times S^n$ is easily seen to be compact, convex, and non-empty. Hence all conditions of Kakutani's fixed point theorem are satisfied and φ has a fixed point

$$\left(x^{*1},\ldots,x^{*m},q^*\right).$$

Clearly

$$x^{*i}\in\tilde{\delta}^i\left(q^*\right)=\delta^i\left(\hat{p}(f(q^*),\underline{p},p),\hat{l}^i\left(f\left(q^*\right),w\right),\hat{L}^i\left(f\left(q^*\right),w\right),w^i\right),\ \forall i\in I_m,\tag{26}$$

and

$$\bar{q}^* \in \mu\left(x^{*1}, \ldots, x^{*m}\right).$$

It will be shown that all the conditions of a constrained equilibrium are satisfied at

$$\left(x^{*1}, \ldots, x^{*m}, \hat{l}\left(f\left(\bar{q}^*\right), w\right), \hat{L}\left(f\left(\bar{q}^*\right), w\right), \hat{p}(f(\bar{q}^*), \underline{p}, \bar{p})\right).$$

By (26) it is implied that Condition 1 of a constrained equilibrium is satisfied. Next Condition 2,

$$\sum_{i=1}^{m} x^{*i} - \sum_{i=1}^{m} w^i = 0 \tag{27}$$

is verified. By the same reasoning as in the proof of Theorem 3.4 it holds that

$$\hat{p}\left(f\left(\bar{q}^*\right), \underline{p}, \bar{p}\right) \cdot \sum_{i=1}^{m} x^{*i} = \hat{p}\left(f\left(\bar{q}^*\right), \underline{p}, \bar{p}\right) \cdot \sum_{i=1}^{m} w^i. \tag{28}$$

Suppose there is some $h \in I_n$ such that

$$\sum_{i=1}^{m} x_h^{*i} - \sum_{i=1}^{m} w_h^i < 0. \tag{29}$$

Since $\hat{p}(f(\bar{q}^*), \underline{p}, \bar{p}) \geq \underline{p} \gg 0$ it follows that there is a $g \in I_n$ such that

$$\sum_{i=1}^{m} x_g^{*i} - \sum_{i=1}^{m} w_g^i > 0.$$

This and (29) imply according to Lemma 4.1 that $\bar{q}_h^* = 0$ which implies that $f_h\left(\bar{q}^*\right) = 0$. Consequently by (26) and Assumption A5, for all $i \in I_m$,

$$x_h^{*i} - w_h^i \geq \hat{l}_h^i\left(f\left(\bar{q}^*\right), w\right) = 0.$$

Then

$$\sum_{i=1}^{m} x_h^{*i} - \sum_{i=1}^{m} w_h^i \geq 0,$$

a contradiction with (29). Hence,

$$\sum_{i=1}^{m} x^{*i} - \sum_{i=1}^{m} w^i \geq 0. \tag{30}$$

All prices being positive, (28) and (30) imply (27). Using Theorem 3.2 the remaining conditions of a constrained equilibrium follow. Define $q^* = f\left(\bar{q}^*\right)$. Then clearly $q^* \leq \alpha$ and for some $k \in I_n$ it holds that $q_k^* = \alpha_k$ by the definition of f.

Q.E.D.

Notice that if $\min\{\alpha_j \mid j \in I_n\} < \frac{1}{3}$ it cannot be guaranteed that for all $i \in I_m$ there exists some $j \in I_n$ such that $\hat{l}^i_j(q) < 0$, for $q \in f(S^n)$. Therefore the original proof of Drèze (1975) of the continuity of the budget correspondence cannot be used to prove the existence of constrained equilibria if $\min\{\alpha_j \mid j \in I_n\} < \frac{1}{3}$.

Recall that a supply constrained equilibrium is defined as a constrained equilibrium such that there is no market with binding demand rationing and there is at least one market without binding supply rationing. The existence of such a constrained equilibrium is easily shown using Theorem 4.2.

Corollary 4.3
There exists a supply constrained equilibrium of the economy $E = (\{X^i, \succeq^i, w^i\}_{i=1}^m, P_{(\underline{p},\bar{p})})$ *with rationing system* $(\tilde{l}, \tilde{L})$ *if Assumptions A1, A2, A3, A4, and A5 are satisfied.*

Proof
Take $\alpha = \frac{1}{2}\iota^n$ and apply Theorem 4.2. Then there exists an extended supply constrained equilibrium induced by some $q^* \in Q$ satisfying $q^* \leq \alpha$ and for some $k \in I_n$, $q^*_k = \frac{1}{2}$. Clearly for all $i \in I_m$ and for all $j \in I_n$, $\hat{L}^i_j(q^*, w) > \sum_{h \neq i} w^h_j$, and for all $i \in I_m$, $\hat{l}^i_k(q^*, w) < -w^i_k$. Hence commodity k is not rationed and all other commodities are not rationed in their demand.

Q.E.D.

5 Existence of Extended Demand Constrained Equilibria

Recent experiences in Eastern Europe make clear that demand constrained equilibria and extended demand constrained equilibria are interesting too. A demand constrained equilibrium is a constrained equilibrium without binding rationing on excess supply, while on one market there is no binding demand rationing. The commodity of this last market can be chosen ex post as an unrationed numeraire commodity. It is possible to generalize the concept of a demand constrained equilibrium. Let a vector $\beta \in Q$ be given. A constrained equilibrium where the psuedo price on each market $j \in I_n$ is greater than or equal to β_j while on at least one market $k \in I_n$ the pseudo-price is equal to β_k is called an extended demand constrained equilibrium with respect to β. The existence of an extended demand constrained equilibria with respect to an arbitrary $\beta \in Q$ is shown in the following theorem.

Theorem 5.1
Let be given the economy $E = (\{X^i, \succeq^i, w^i\}_{i=1}^m, P_{(\underline{p},\bar{p})})$ *with rationing system* $(\tilde{l}, \tilde{L})$ *and let the Assumptions A1, A2, A3, A4, and A5 be satisfied. Then for every* $\beta \in Q$ *there exists an extended demand constrained equilibrium with respect to* β,

$$\left(x^{*1}, \ldots, x^{*m}, \hat{l}(q^*, w), \hat{L}(q^*, w), \hat{p}(q^*, \underline{p}, \bar{p})\right)$$

of the economy E *with rationing system* $(\tilde{l}, \tilde{L})$, *i.e.* $q^* \in Q$ *is such that* $q^* \geq \beta$ *and for some* $k \in I_n$, $q^*_k = \beta_k$.

Proof

Define the component $j \in I_n$ of the function $f : S^n \to Q$ by

$$f_j(\bar{q}) = 1 - \frac{(1-\beta_j)\,\bar{q}_j}{\max\{\bar{q}_1,\ldots,\bar{q}_n\}},\ \forall \bar{q} \in S^n.$$

Clearly f is a continuous function on S^n. Define the correspondence $\tilde{\delta}^i : S^n \to \bar{X}^i$ by $\tilde{\delta}^i(\bar{q}) = \bar{\delta}^i(f(\bar{q}))$, where $\bar{\delta}^i$ is the correspondence defined in the proof of Theorem 3.4. Since f is continuous on S^n and $\bar{\delta}^i$ is upper semi-continuous on Q it follows that $\tilde{\delta}^i$ is upper semi-continuous on S^n. Define the correspondence ν from $\prod_{i=1}^m \bar{X}^i$ into S^n by

$$\nu(x^1,\ldots,x^m) = \left\{ q \in S^n \,\middle|\, q \cdot \sum_{i=1}^m (x^i - w^i) \le \bar{q} \cdot \sum_{i=1}^m (x^i - w^i),\ \forall \bar{q} \in S^n \right\},\ \forall (x^1,\ldots,x^m) \in \prod_{i=1}^m \bar{X}^i.$$

In a similar way as in the proof of Lemma 4.1 it can be shown that ν is non-empty and convex valued, and upper semi-continuous. Moreover, if $(x^1,\ldots,x^m) \in \prod_{i=1}^m \bar{X}^i$ and for some $g, h \in I_n$ it holds that $\sum_{i=1}^m x_g^i - \sum_{i=1}^m w_g^i > \sum_{i=1}^m x_h^i - w_h^i$, then $q_g = 0$, for all $q \in \nu(x^1,\ldots,x^m)$. Define the correspondence $\varphi : \prod_{i=1}^m \bar{X}^i \times S^n \to \prod_{i=1}^m \bar{X}^i \times S^n$ by

$$\varphi\left(x^1,\ldots,x^m,\bar{q}\right) = \prod_{i=1}^m \tilde{\delta}^i(\bar{q}) \times \nu\left(x^1,\ldots,x^m\right),\ \forall \left(x^1,\ldots,x^m,\bar{q}\right) \in \prod_{i=1}^m \bar{X}^i \times S^n.$$

This correspondence is upper semi-continuous on $\prod_{i=1}^m \bar{X}^i \times S^n$ by the remarks above and the upper semi-continuity of $\tilde{\delta}^i$, $\forall i \in I_m$. Using the same argument as in the proof of Theorem 4.2 it is shown that for all $(x^1,\ldots,x^m,\bar{q}) \in \prod_{i=1}^m \bar{X}^i \times S^n$ the set $\prod_{i=1}^m \tilde{\delta}^i(\bar{q}) \times \nu(x^1,\ldots,x^m)$ is convex. Again all conditions of Kakutani's fixed point theorem are satisfied and φ has a fixed point $(x^{*1},\ldots,x^{*m},\bar{q}^*)$. Clearly

$$x^{*i} \in \tilde{\delta}^i(\bar{q}^*) = \delta^i\left(\hat{p}(f(\bar{q}^*),\underline{p},\bar{p}), \hat{l}^i(f(\bar{q}^*),w), \hat{L}^i(f(\bar{q}^*),w), w^i\right),\ \forall i \in I_m, \tag{31}$$

and

$$\bar{q}^* \in \nu\left(x^{*1},\ldots,x^{*m}\right).$$

It will be shown that all the conditions of a constrained equilibrium are satisfied at

$$\left(x^{*1},\ldots,x^{*m}, \hat{l}(f(\bar{q}^*),w), \hat{L}(f(\bar{q}^*),w), \hat{p}(f(\bar{q}^*),\underline{p},\bar{p})\right).$$

By (31) it is implied that Condition 1 of a constrained equilibrium is satisfied. Next Condition 2,

$$\sum_{i=1}^m \left(x^{*i} - w^i\right) = 0 \tag{32}$$

is verified. By the same reasoning as in the proof of Theorem 4.2 it holds that

$$\hat{p}\left(f(\bar{q}^*),\underline{p},\bar{p}\right) \cdot \sum_{i=1}^m x^{*i} = \hat{p}\left(f(\bar{q}^*),\underline{p},\bar{p}\right) \cdot \sum_{i=1}^m w^i. \tag{33}$$

Suppose there is some $h \in I_n$ such that $\sum_{i=1}^{m} x_h^{*i} - \sum_{i=1}^{m} w_h^i < 0$. Since $\hat{p}(f(\bar{q}^*), \underline{p}, \bar{p}) \geq \underline{p} \gg 0$ it follows that there is a $g \in I_n$ such that

$$\sum_{i=1}^{m} x_g^{*i} - \sum_{i=1}^{m} w_g^i > 0. \tag{34}$$

This implies according to the remarks made above that $\bar{q}_g^* = 0$ which implies that $f_g(\bar{q}^*) = 1$. Consequently by Assumption A5, for all $i \in I_m$, $x_g^{*i} - w_g^i \leq \hat{L}_g^i(f(\bar{q}^*), w) = 0$. Then $\sum_{i=1}^{m} x_g^{*i} - \sum_{i=1}^{m} w_g^i \leq 0$, a contradiction with (34). Hence,

$$\sum_{i=1}^{m} x^{*i} - \sum_{i=1}^{m} w^i \geq 0. \tag{35}$$

All prices being positive, (33) and (35) imply (32). Using Theorem 3.2 the remaining conditions of a constrained equilibrium follow. Define $q^* = f(\bar{q}^*)$. Then clearly $q^* \geq \beta$ and for some $k \in I_n$ it holds that $q_k^* = \beta_k$ by the definition of f.

Q.E.D.

Notice that if $\max\{\beta_j \mid j \in I_n\} < \frac{1}{3}$ it cannot be guaranteed that for all $i \in I_m$ there exists some $j \in I_n$ such that $\hat{l}_j^i(q) < 0$ for all $q \in f(S^n)$. In order to prove the existence of a constrained equilibrium for these values of β the proof of the continuity of the budget correspondence of Drèze (1975) cannot be used.

Since a demand constrained equilibrium is defined as a constrained equilibrium such that there is no market with binding supply rationing and there is at least one market without binding demand rationing, the existence of such a constrained equilibrium is easily shown using Theorem 5.1.

Corollary 5.2
There exists a demand constrained equilibrium of the economy $E = (\{X^i, \succeq^i, w^i\}_{i=1}^{m}, P_{(\underline{p},\bar{p})})$ *with rationing system* $(\tilde{l}, \tilde{L})$ *if Assumptions A1, A2, A3, A4, and A5 are satisfied.*

Proof
Take $\beta = \frac{1}{2}\iota^n$ and apply Theorem 5.1. Then there exists an extended demand constrained equilibrium induced by some $q^* \in Q$ satisfying $q^* \geq \beta$ and for some $k \in I_n$, $q_k^* = \frac{1}{2}$. Clearly for all $i \in I_m$ and for all $j \in I_n$, $\hat{l}_j^i(q^*, w) < -w_j^i$, and for all $i \in I_m$, $\hat{L}_k^i(q^*, w) > \sum_{h \neq i} w_k^h$. Hence commodity k is not rationed and all other commodities are not rationed in their supply.

Q.E.D.

It should be noted that all possible constrained equilibria are obtained by the correspondences used in the Theorems 3.4, 4.2, and 5.1, whose fixed points are constrained equilibria. By Theorem 3.3, every constrained equilibrium for some economy E with rationing system

$(\tilde{l}, \tilde{L})$ can be written as

$$\left(x^{*1}, \ldots, x^{*m}, \hat{l}(q^*, w), \hat{L}(q^*, w), \hat{p}(q^*, \underline{p}, \bar{p})\right)$$

for some $q^* \in Q$. Such a constrained equilibrium satisfies the conditions of Theorem 3.4 for every $k \in I_n$ and λ equal to q_k^*. It satisfies the Conditions of Theorem 4.2 for every $\alpha \in Q$ satisfying $\alpha \geq q^*$ and for some $k \in I_n$, $\alpha_k = q_k^*$. It also satisfies the Conditions of Theorem 5.1 for every $\beta \in Q$ satisfying $\beta \leq q^*$ and for some $k \in I_n$, $\beta_k = q_k^*$.

6 The Upper Semi-Continuity of the Equilibrium Correspondence

For all consumers $i \in I_m$, suppose the consumption set X^i and the preference relation $\succeq^i$ is given. Moreover suppose the rationing system $(\tilde{l}, \tilde{L})$ is given. In this section it is shown that the equilibrium correspondence, which assigns to every specification of initial endowments and set of admissible prices the set of all constrained equilibrium allocations, is upper semi-continuous. Define the equilibrium correspondence $\mathcal{E} : R \times \prod_{i=1}^m \mathrm{Int}\,(X^i)$ by

$$\begin{aligned}\mathcal{E}(r, w) \;=\; &\Big\{(x^1, \ldots, x^m)\Big|\exists q \in Q,\ \exists p \in P_r \text{ such that } (x^1, \ldots, x^m, \hat{l}(q, w), \hat{L}(q, w), p)\\ &\text{is a constrained equilibrium for the economy } E = (\{X^i, \succeq^i, w^i\}_{i=1}^m, P_r)\\ &\text{with rationing system } (\tilde{l}, \tilde{L})\Big\},\ \forall (r, w) \in R \times \prod_{i=1}^m \mathrm{Int}(X^i).\end{aligned}$$

By Theorem 3.3 it is guaranteed that $\mathcal{E}(r, w)$ contains all constrained equilibrium allocations for the set of admissible prices P_r and initial endowments w. In order to prove the upper semi-continuity of the equilibrium correspondence, one additional assumption has to be made.

A6. The functions $\tilde{l} : Q \times \prod_{i=1}^m \mathrm{Int}\,(X^i) \to -\mathbb{R}_+^{mn}$ and $\tilde{L} : Q \times \prod_{i=1}^m \mathrm{Int}\,(X^i) \to \mathbb{R}_+^{mn}$ are continuous.

Theorem 6.1

For all $i \in I_m$ let the consumption set X^i satisfy Assumption A1, the preference relation $\succeq^i$ satisfy Assumption A2, and let the rationing system $(\tilde{l}, \tilde{L})$ satisfy Assumptions A5 and A6. Then the correspondence $\mathcal{E} : R \times \prod_{i=1}^m Int\,(X^i) \to \prod_{i=1}^m X^i$ is upper semi-continuous.

Proof

It will be shown that $\mathcal{E}$ is upper semi-continuous at every point $(r, w) \in R \times \prod_{i=1}^m \mathrm{Int}\,(X^i)$. By Theorem 3.4, 4.2, or 5.1, $\mathcal{E}(r, w) \neq \emptyset$. Moreover, $\mathcal{E}$ is easily seen to be compact valued. Let $((r^s, w^s))_{s \in \mathbb{N}}$ be a sequence in $R \times \prod_{i=1}^m \mathrm{Int}\,(X^i)$ converging to (r, w). Let $((x^{1^s}, \ldots, x^{m^s}))_{s \in \mathbb{N}}$ be a sequence with $(x^{1^s}, \ldots, x^{m^s}) \in \mathcal{E}(r^s, w^s)$. It has to be shown that there is a converging subsequence of the sequence $((x^{1^s}, \ldots, x^{m^s}))_{s \in \mathbb{N}}$ whose limit belongs to $\mathcal{E}(r, w)$. For every $s \in N$ there is a $q^s \in Q$ such that

$$\left(x^{1^s}, \ldots, x^{m^s}, \hat{l}(q^s, w^s), \hat{L}(q^s, w^s), \hat{p}(q^s, r^s)\right)$$

is a constrained equilibrium for the economy $E = (\{X^i, \succeq^i, w^{i^s}\}_{i=1}^m, P_{r^s})$ with rationing system $(\tilde{l}, \tilde{L})$. Consider the sequence $((x^{1^s}, \ldots, x^{m^s}, q^s))_{s\in\mathbf{N}}$. Since Q is bounded the sequence $(q^s)_{s\in\mathbf{N}}$ is bounded. For all $h \in I_m$, since

$$0 \leq x^{h^s} \leq \sum_{i=1}^m w^{i^s} \to \sum_{i=1}^m w^i,$$

it follows also that the sequence $(x^{h^s})_{s\in\mathbf{N}}$ is bounded. Therefore the sequence $((x^{1^s}, \ldots, x^{m^s}, q^s))_{s\in\mathbf{N}}$ has a convergent subsequence, $((x^{1^{s^t}}, \ldots, x^{m^{s^t}}, q^{s^t}))_{t\in\mathbf{N}}$ with limit, say $(x^1, \ldots, x^m, q)$. It will be shown that

$$\left(x^1, \ldots, x^m, \hat{l}(q,w), \hat{L}(q,w), \hat{p}(q,r)\right)$$

satisfies all the conditions of a constrained equilibrium for the economy $E = (\{X^i, \succeq^i, w^i\}_{i=1}^m, P_r)$ with rationing system $(\tilde{l}, \tilde{L})$. By Assumptions A1 and A5 it is obvious that

$$\left(x^1, \ldots, x^m, \hat{l}(q,w), \hat{L}(q,w), \hat{p}(q,r)\right) \in \prod_{i=1}^m X^i \times \prod_{i=1}^m -\mathbf{R}_+^n \times \prod_{i=1}^m \mathbf{R}_+^n \times P_r.$$

Using Theorem 2.2 and the maximum theorem it follows that every δ^i is upper semi-continuous on $R \times -\mathbf{R}_+^n \times \mathbf{R}_+^n \times \mathrm{Int}\,(X^i)$. Using Assumption A6 it is easily seen that $\hat{l}$ is a continuous function from $Q \times \prod_{i=1}^m \mathrm{Int}\,(X^i)$ into $-\mathbf{R}_+^{mn}$ and $\hat{L}$ is a continuous function from $Q \times \prod_{i=1}^m \mathrm{Int}\,(X^i)$ into $\mathbf{R}_+^{mn}$. Moreover $\hat{p}$ is a continuous function from $Q \times R$ into $\mathbf{R}_{++}^n$. Therefore for every $i \in I_m$ the correspondence $\bar{\delta}^i : Q \times R \times \prod_{i=1}^m \mathrm{Int}\,(X^i)$ into X^i defined by

$$\bar{\delta}^i(q,r,w) = \delta^i\left(\hat{p}(q,r), \hat{l}^i(q,w), \hat{L}^i(q,w), w^i\right), \ \forall (q,r,w) \in Q \times R \times \prod_{i=1}^m \mathrm{Int}\left(X^i\right),$$

is upper semi-continuous on $Q \times R \times \prod_{i=1}^m \mathrm{Int}\,(X^i)$. Since $(q^{s^t}, r^{s^t}, w^{s^t}) \to (q,r,w)$ and $x^{i^{s^t}} \to x^i$ it holds by the upper semi-continuity of $\bar{\delta}^i$ that

$$x^i \in \bar{\delta}^i(q,r,w) = \delta^i\left(\hat{p}(q,r), \hat{l}^i(q,w), \hat{L}^i(q,w), w^i\right), \ \forall i \in I_m.$$

So Condition 1 of a constrained equilibrium is satisfied. Clearly $\sum_{i=1}^m x^i = \lim_{t\to\infty} \sum_{i=1}^m x^{i^{s^t}} = \lim_{t\to\infty} \sum_{i=1}^m w^{i^{s^t}} = \sum_{i=1}^m w^i$, thereby giving Condition 2. By Theorem 3.2 the remaining conditions of a constrained equilibrium are satisfied.

Q.E.D.

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